RELATIVITY AND COSMOLOGY I

Solutions to Problem Set 8 Fall 2023

1. Gravitational Redshift

(a) Since the geometry is static, so are geodesics. If the first photon took T_1 time in these coordinates to go from Alice to Bob, the second photon will take the same amount of time.

(b) We are evaluating the proper time of two events that happened at the same spatial point. We thus need to compute

$$d\tau^2(\partial_t, \partial_t) = -ds^2(\partial_t, \partial_t) = (1 + 2\Phi(x, y, z))\mathrm{d}t^2 \,. \tag{1}$$

The finite interval will thus be given by

$$\Delta\tau_A = \int d\tau = \sqrt{1 + 2\Phi(\vec{x_A})}\Delta t \,. \tag{2}$$

(c) Analogously, for Bob

$$\Delta\tau_B = \int d\tau = \sqrt{1 + 2\Phi(\vec{x_B})}\Delta t \,. \tag{3}$$

(d) Comparing them, we get

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \sqrt{\frac{1 + 2\Phi(\vec{x}_B)}{1 + 2\Phi(\vec{x}_A)}} \approx \sqrt{1 + 2\left(\Phi(\vec{x}_B) - \Phi(\vec{x}_A)\right)} \approx 1 + \Phi(\vec{x}_B) - \Phi(\vec{x}_A) \,. \tag{4}$$

That means that if Alice and Bob measure this interval of time in their respective reference frames, they will see a difference proportional to the gradient of the gravitational field between them.

(e) The electromagnetic wave produced by Alice would have wavelength $\lambda_A = T_A$ (remember that $c = 1$). The wavelength of the light received by Bob would be

$$\lambda_B \approx (1 + \Phi(\vec{x}_B) - \Phi(\vec{x}_A))\lambda_A \,. \tag{5}$$

To interpret this answer, consider the case in which the two observers are orbiting around a planet. The Newtonian protential for this system is $\Phi(\vec{x}) = -\frac{G_N M}{r}$. The formula becomes

$$\frac{\lambda_B}{\lambda_A} \approx 1 + GM\frac{r_B - r_A}{r_A r_B} \,. \tag{6}$$

If $r_A > r_B$ Bob observes a blueshift in the light, if $r_A < r_B$ he observes a redshift.

(f) As there is no explicit dependence on time in metric components, $\partial_t g = 0$, shifting the coordinates by t is indeed an isometry.

(g) The geodesic equations (for an affine parametrisation) are derived from the functional variations of

$$S[t,x,y,z] = \int d\lambda \; g_{\mu\nu}\dot{x^\mu}\dot{x^\nu}. \tag{7}$$

In classical mechanics with coordinates q, if the Lagrangian $L(q,\dot{q})$ does not depend explicitely in q, then $\frac{\partial L}{\partial \dot{q}}$ is a constant of motion, as follows from the Euler-Lagrange equations. Here exactly the same idea applies and $L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ does not depend on t, hence

$$\frac{\partial L}{\partial \dot{t}} = 2g_{\mu t}\dot{x}^\mu = 2g_{tt}\dot{t} \tag{8}$$

is a constant of motion, i.e.

$$\frac{d}{d\lambda}\left((1+2\Phi)\frac{dt}{d\lambda}\right) = 0 \; . \tag{9}$$

Alternatively, the geodesic equation for coordinate t is

$$\frac{d^2t}{d\lambda^2} + 2\Gamma^t_{it}\frac{dx^i}{d\lambda}\frac{dt}{d\lambda} = 0 \; ,$$

with $i = 1,2,3$. Expanding the Christoffel $\Gamma^t_{it} = \frac{\partial_i\Phi}{1+2\Phi}$ one can write

$$(1+2\Phi)\frac{d^2t}{d\lambda^2} + 2\partial_i\Phi\frac{dx^i}{d\lambda}\frac{dt}{d\lambda} = 0 \; ,$$

Using the chain rule $\partial_i\Phi\frac{dx^i}{d\lambda} = \frac{d\Phi}{d\lambda}$ and product rule, this is equivalent to

$$\frac{d}{d\lambda}\left((1+2\Phi)\frac{dt}{d\lambda}\right) = 0 \; . \tag{10}$$

This implies $\varepsilon = g_{tt}V^t = (1+2\Phi)\frac{dt}{d\lambda}$ is a constant of motion.

Another way to show it is by using the covariant derivative expression of Geodesics. For an affinely parametrized geodesic $X^\mu(\lambda)$, we have that $V^\mu = \frac{dX^\mu}{d\lambda}$ satisfies :

$$V^\mu D_\mu(V^\nu) = 0 \tag{11}$$

Let us now write $\epsilon = V_\mu\xi^\mu$ where $\xi^\mu = \delta^\mu_t$ are the components of the killing vector ∂_t. By definition, the Killing vectors satisfies $D_\mu\xi_\nu + D_\nu\xi_\mu = 0$. Putting it all together :

$$V^\mu D_\mu(V^\nu\xi_\nu) = V^\mu D_\mu(V^\nu)\xi_\nu + V^\mu V^\nu D_\mu(\xi_\nu) \tag{12}$$

The first term vanishes by the affine geodesic equation, the second by the killing equation. By $V^\mu D_\mu \equiv \frac{d}{d\lambda}$ (chain rule), we have the result.

(h) The four-velocities of Alice and Bob are $U_A = \frac{1}{\sqrt{1+2\Phi(x_A)}}\partial_t$ and $U_B = \frac{1}{\sqrt{1+2\Phi(x_A)}}\partial_t$ respectively. The dt component of conjugate momentum of photon is $p_t = g_{tt}V^t = \varepsilon$. Therefore,

$$E_A = p_t U^t_A = \frac{\varepsilon}{\sqrt{1+2\Phi(x_A)}}$$

$$E_B = p_t U^t_B = \frac{\varepsilon}{\sqrt{1+2\Phi(x_B)}}$$

Eliminating ε among these equations give

$$E_B = \sqrt{\frac{1+2\Phi(x_A)}{1+2\Phi(x_B)}}E_A \tag{13}$$

or in terms of wavelength they measure,

$$\lambda_B = \sqrt{\frac{1+2\Phi(x_B)}{1+2\Phi(x_A)}}\lambda_A \tag{14}$$

The energy relation holds for any particle, not only photons.

2. The Bianchi Identity

(a) We prove this in reverse. Start from

$$\begin{aligned} \nabla_{[\tau}R_{\mu\nu]\rho\sigma} &= 0 \\ \nabla_\tau R_{\mu\nu\rho\sigma} - \nabla_\mu R_{\tau\nu\rho\sigma} + \nabla_\mu R_{\nu\tau\rho\sigma} - \nabla_\tau R_{\nu\mu\rho\sigma} - \nabla_\nu R_{\mu\tau\rho\sigma} + \nabla_\nu R_{\tau\mu\rho\sigma} &= 0 \\ \nabla_\tau R_{\rho\sigma\mu\nu} + \nabla_\mu R_{\rho\sigma\nu\tau} + \nabla_\mu R_{\rho\sigma\nu\tau} + \nabla_\tau R_{\rho\sigma\mu\nu} + \nabla_\nu R_{\mu\tau\rho\sigma} + \nabla_\nu R_{\rho\sigma\tau\mu} &= 0 \\ \nabla_\tau R_{\rho\sigma\mu\nu} + \nabla_\mu R_{\rho\sigma\nu\tau} + \nabla_\nu R_{\mu\tau\rho\sigma} &= 0\,, \end{aligned} \tag{15}$$

where from the second to the third line we used the symmetries of the Riemann tensor to write every term with the indices in the orders that appear on the Problem Set.

(b) The first identity is nothing else than the definition of the Riemann tensor from the action of the commutator of covariant derivatives acting on a $(0,2)$ tensor. In general,

$$[\nabla_\rho, \nabla_\sigma]A_{\mu\nu} = -R^\lambda{}_{\mu\rho\sigma}A_{\lambda\nu} - R^\lambda{}_{\nu\rho\sigma}A_{\mu\lambda}\,, \tag{16}$$

which explains the equation. For the second one, we act with a covariant derivative on the action of the Riemann tensor on a dual vector

$$\begin{aligned} \nabla_\rho[\nabla_\sigma, \nabla_\mu]V_\nu &= \nabla_\rho(-R^\lambda{}_{\nu\sigma\mu}V_\lambda) \\ &= -V_\lambda\nabla_\rho R^\lambda{}_{\nu\sigma\mu} - R^\lambda{}_{\nu\sigma\mu}\nabla_\rho V_\lambda\,. \end{aligned} \tag{17}$$

(c) This is true simply because on both sides of the equation we are subtracting $[\rho,\sigma,\mu]$ and the antisymmetrization of one of its odd permutations. Infact,

$$D_{[\rho\sigma\mu]} - D_{[\sigma\rho\mu]} = 2D_{[\rho\sigma\mu]}\,. \tag{18}$$

(d) Consider that, an equivalent way to write

$$\nabla_{[\rho}\nabla_\sigma\nabla_{\mu]}V_\nu - \nabla_{[\sigma}\nabla_\rho\nabla_{\mu]}V_\nu = \nabla_{[\rho}\nabla_\sigma\nabla_{\mu]}V_\nu - \nabla_{[\rho}\nabla_\mu\nabla_{\sigma]}V_\nu \tag{19}$$

is

$$[\nabla_{[\rho}, \nabla_\sigma]\nabla_{\mu]}V_\nu = \nabla_{[\rho}[\nabla_\sigma, \nabla_{\mu]}]V_\nu \tag{20}$$

This is nothing else than the **Jacobi identity**. Applying the formulas from before, we get that it is equivalent to

$$-R^\lambda_{[\mu\rho\sigma]}\nabla_\lambda V_\nu - R^\lambda{}_{\nu[\rho\sigma}\nabla_{\mu]}V_\lambda = -V_\lambda\nabla_{[\rho}R^\lambda{}_{|\nu|\sigma\mu]} - R^\lambda{}_{\nu[\sigma\mu}\nabla_{\rho]}V_\lambda\,, \tag{21}$$

which is the equation that we wanted.

(e) The first term on the left hand side of (21) vanishes by the symmetries of the Riemann tensor. The second term on the left hand side of (21) exactly cancels the second term on its right hand side. We are left with

$$V_\lambda \nabla_{[\rho} R^\lambda{}_{|\nu|\sigma\mu]} = 0\,. \tag{22}$$

We can write

$$V_\lambda \nabla_{[\rho} R^\lambda{}_{|\nu|\sigma\mu]} = V^\lambda \nabla_{[\rho} R_{|\lambda\nu|\sigma\mu]} = V^\lambda \nabla_{[\rho} R_{\sigma\mu]\lambda\nu}\,, \tag{23}$$

so that (22) for a generic vector V implies

$$\nabla_{[\rho} R_{\sigma\mu]\lambda\nu} = 0\,, \tag{24}$$

which we had argued is equivalent to the Bianchi identity.

(f) Start from the original form of the Bianchi identity

$$\nabla_\tau R_{\rho\sigma\mu\nu} + \nabla_\mu R_{\rho\sigma\nu\tau} + \nabla_\nu R_{\rho\sigma\tau\mu} = 0\,. \tag{25}$$

We take two traces

$$\begin{aligned} g^{\sigma\nu} g^{\rho\tau} (\nabla_\tau R_{\rho\sigma\mu\nu} + \nabla_\mu R_{\rho\sigma\nu\tau} + \nabla_\nu R_{\rho\sigma\tau\mu}) &= 0 \\ \nabla^\rho R_{\rho\mu} - \nabla_\mu R + \nabla^\sigma R_{\sigma\mu} &= 0 \\ \nabla^\rho G_{\rho\mu} &= 0\,, \end{aligned} \tag{26}$$

as we wanted to prove.

3. The Coriolis Force

(a) By the chain rule, the velocity $u^i = \frac{dx^i}{dt}$ can be expressed as

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt}\,. \tag{27}$$

The trajectory $\ddot{x}^\mu = 0$ implies

$$\frac{dt}{d\lambda} = c_t\,, \qquad \frac{dx^i}{d\lambda} = c_i\,, \tag{28}$$

where c_t and c_i are constants. This makes the velocity

$$\frac{dx^i}{dt} = \frac{c_i}{c_t} \tag{29}$$

constant as well.

(b) We use the Euler-Lagrange equations

$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu}\,. \tag{30}$$

The right hand side is trivially 0, as the Lagrangian does not explicitly depend on position. The left hand side gives

$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{d}{d\lambda}\frac{\partial\sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}{\partial \dot{x}^\mu} = \frac{d}{d\lambda}\left(\frac{1}{2\sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}\frac{-\eta_{\gamma\delta}\partial\left(\dot{x}^\gamma\dot{x}^\delta\right)}{\partial\dot{x}^\mu}\right) = \frac{d}{d\lambda}\left(\frac{-\eta_{\mu\gamma}\dot{x}^\gamma}{\sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}\right) \tag{31}$$

As $\ddot{x}^\mu = 0$, the expression in last brackets is constant, so $\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = 0$.

Alternatively, since the Lagrangian does not depend on x^μ, it is a cyclic variable, and so by Noether's theorem $\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}$ is a constant.

(c) Let's start by applying the chain rule $\frac{dx^\mu}{d\lambda} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{dx^{\mu'}}{d\lambda}$ twice

$$\ddot{x}^\mu = \frac{d}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = \frac{d}{d\lambda}\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{dx^{\mu'}}{d\lambda}\right) = \frac{dx^{\mu'}}{d\lambda}\frac{d}{d\lambda}\frac{\partial x^\mu}{\partial x^{\mu'}} + \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{d^2x^{\mu'}}{d\lambda^2} = \tag{32}$$

$$= \frac{dx^{\mu'}}{d\lambda}\frac{dx^{\nu'}}{d\lambda}\frac{\partial^2 x^\mu}{\partial x^{\mu'}\partial x^{\nu'}} + \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{d^2x^{\mu'}}{d\lambda^2} = \dot{x}^{\mu'}\dot{x}^{\nu'}\frac{\partial^2 x^\mu}{\partial x^{\mu'}\partial x^{\nu'}} + \frac{\partial x^\mu}{\partial x^{\mu'}}\ddot{x}^{\mu'} \tag{33}$$

Therefore, by multiplying geodesic equation $0 = \ddot{x}^\mu$ by $\frac{\partial x^{\mu'}}{\partial x^\mu}$ one gets

$$\frac{\partial x^{\mu'}}{\partial x^\mu}\ddot{x}^\mu = \ddot{x}^{\mu'} + \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial x^{\nu'}\partial x^{\lambda'}}\dot{x}^{\nu'}\dot{x}^{\lambda'} = \ddot{x}^{\mu'} + \Gamma^{\mu'}_{\nu'\lambda'}\dot{x}^{\nu'}\dot{x}^{\lambda'} \tag{34}$$

with $\Gamma^{\mu'}_{\nu'\lambda'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial x^{\nu'}\partial x^{\lambda'}}$. Note that this is the same value one gets from computing Christoffel symbols for $g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\eta_{\mu\nu}$

(d) From the transformation laws, we find how differentials transform

$$\begin{cases} \mathrm{d}t = \mathrm{d}t' \\ \mathrm{d}x = \cos(\omega t')\mathrm{d}x' + \sin(\omega t')\mathrm{d}y' + \omega\left(y'\cos(\omega t') - x'\sin(\omega t')\right)\mathrm{d}t' \\ \mathrm{d}y = -\sin(\omega t')\mathrm{d}x' + \cos(\omega t')\mathrm{d}y' - \omega\left(x'\cos(\omega t') + y'\sin(\omega t')\right)\mathrm{d}t' \\ \mathrm{d}z = \mathrm{d}z'\,. \end{cases} \tag{35}$$

The line element in the new rotating system of reference is thus

$$ds^2 = -(1-\omega^2(x'^2+y'^2))\mathrm{d}t'^2 + \omega y'(\mathrm{d}t'\mathrm{d}x' + \mathrm{d}x'\mathrm{d}t') - \omega x'(\mathrm{d}t'\mathrm{d}y' + \mathrm{d}y'\mathrm{d}t') + \mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2\,. \tag{36}$$

(e) To find the Christoffel symbols we use what we found

$$\Gamma^{\mu'}_{\nu'\lambda'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial x^{\nu'}\partial x^{\lambda'}}\,. \tag{37}$$

It will help to consider that the inverse transformation is given by a rotation by an angle of opposite sign

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} t \\ x\cos(\omega t) - y\sin(\omega t) \\ x\sin(\omega t) + y\cos(\omega t) \\ z \end{pmatrix}\,. \tag{38}$$

By staring at (37) it should become clear that $\Gamma^{t'}_{\mu'\nu'} = \Gamma^{z'}_{\mu'\nu'} = 0$ for all sets of lower indices. The non-vanishing terms are

$$\begin{aligned}
\Gamma^{x'}_{t't'} &= -x'\omega^2 \,, \\
\Gamma^{y'}_{t't'} &= -y'\omega^2 \,, \\
\Gamma^{x'}_{y't'} &= \omega \,, \\
\Gamma^{y'}_{x't'} &= -\omega \,.
\end{aligned} \tag{39}$$

The equations of motion become

$$\begin{cases}
\ddot{t}' = 0 \\
\ddot{x}' - x'\omega^2\dot{t}'^2 + 2\omega\dot{y}'\dot{t}' = 0 \\
\ddot{y}' - y'\omega^2\dot{t}'^2 - 2\omega\dot{t}'\dot{x}' = 0 \\
\ddot{z}' = 0
\end{cases} \tag{40}$$

From the first equation we get $t'(\lambda) = a\lambda + b$. Using the chain rule, we can cast the equations of motion for x' and y' in terms of derivatives with respect to t'.

$$\begin{cases}
\frac{d}{d\lambda}\left(\dot{t}'\frac{dx'}{dt}\right) - x'\omega^2 a^2 + 2\omega\dot{t}'\frac{dy'}{dt}a = 0 \\
\frac{d}{d\lambda}\left(\dot{t}'\frac{dy'}{dt}\right) - y'\omega^2 a^2 - 2\omega\dot{t}'\frac{dx'}{dt}a = 0
\end{cases} \tag{41}$$

which reduce to

$$\begin{cases}
\frac{d^2x'}{dt'^2} = \omega^2 x' - 2\omega\frac{dy'}{dt'} \\
\frac{d^2y'}{dt'^2} = \omega^2 y' + 2\omega\frac{dx'}{dt'}
\end{cases} \tag{42}$$

The first term in each equation represents a force that pushes the particle away from the center of the reference frame, and grows with its radial distance from the center: that is the centrifugal force. The remaining terms are proportional to the perpendicular velocity: they represent the Coriolis force. As you can see, in GR all the fictituous forces that appear in non-inertial frames are encoded in the geometry of spacetime.